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Optimal Ritz Vectors for Component Mode Synthesis Using the Singular Value Decomposition

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A fundamental property of the Rayleigh–Ritz method is that a reduced model gives exact predictions if the solutions lie in the range of the reduction basis. For cases with and without rigid-body modes, it is shown that the strain energy norm can be used to define optimal projections of both loads and displacements onto the exactly represented subspace of a given reduced model. In practice, multiple loads or displacements are considered, and variants of the singular value decomposition based on the strain and kinetic energy norms are shown to provide ways to select important contributions. The proposed framework is used to define two new methods for optimal reduction and reduced model correction. The validity and usefulness of the proposed methods are illustrated for the example of two square plates assembled at a right angle with two very different plate thickness configurations.

Nomenclature

$[b]_{N\times NS}$	= frequency-independent input shape matrix
$[b_M]_{N\times NP}$	= matrix of the NP imbalance loads
$[C]_{NS\times N}$	= frequency-independent output shape matrix
$[M]$, $[C]$, $[K]$	= full-order model mass, damping, and stiffness matrices
N	= size of the full-order model
NA	= number of considered inputs
NP	= number of imbalance loads
NR	= size of a reduced-order model
NS	= number of considered outputs
$\{q\}_{N\times 1}$	= degrees of freedom (DOFs) of the full-order model
$\{q_R\}_{NR\times 1}$	= DOFs of a reduced model
$[S]$	= diagonal matrix of singular values
s	= Laplace's variable, $j\omega$
$[T]$	= reduction basis (displacement vectors)
$[U]$	= right singular vectors
$\{u\}_{NA\times 1}$	= force inputs
$[V]$	= left singular vectors
$\{y\}_{NS\times 1}$	= displacement outputs
α	= maximum relative frequency error
π	= projection onto a static load/displacement subspace
$[\phi]$	= matrix of normal modes
$[\Omega^2]$	= diagonal modal stiffness matrix (modal frequencies squared ω_j^2)

I. Introduction

FOR an increasing number of systems, highly detailed component models are needed for accurate component predictions and particularly for stress analysis. For system predictions, it is generally not economically possible to couple the detailed component models, and in many cases, a simplified finite element model of the component is introduced (with approximations on the geometry, the properties, etc.). An efficient alternative is to mathematically reduce the order of the detailed component model. This has led to numerous substructuring, model reduction, superelement, or component mode synthesis (CMS) methods (see the review in Ref. 1, for example).

In most cases model reduction is based on a Rayleigh–Ritz procedure where an approximate solution is found within a finite dimensional vector space that verifies kinematic boundary conditions. As shown in the review done in Sec. II, bases are generally constructed by retaining solutions to particular subproblems that are easily computed or already available (static responses, normal modes of components, etc.).

A fundamental property of the Rayleigh–Ritz method is that a reduced model gives exact predictions if the solutions lie in the range of the reduction basis. Provided that a norm is defined on the vector space of possible solutions, one can easily construct an optimal projection of any displacement onto the exactly represented subspace and use the norm of the residual as a measure of error. Based on these considerations, a new framework for the analysis of reduced models is introduced in Sec. III. It is shown that the strain energy norm can be used to obtain optimal static projections of both displacements and loads, that the singular value decomposition (SVD) allows a treatment of cases with multiple displacements or loads, and that rigid-body modes that have no strain energy can be handled.

Two original methods based on the proposed framework are detailed in Sec. IV. Optimal reductions are obtained using rank constrained projections resulting from the SVD of a given reduction basis. Corrections to a given reduced model are obtained through a decomposition of imbalance loads linked to predicted normal modes. The efficiency of these methods is illustrated in Sec. V using the example of two square plates assembled at a right angle with two very different plate thickness configurations.

II. Displacement-Based Model Reduction

This study considers cases where the structure has an accurate second-order representation, generally constructed using the finite element method, of the form

$$[Ms^2 + Cs + K]_{N\times N}\{q\}_{N\times 1} = [b]_{N\times NA}\{u\}_{NA\times 1}$$
$$\{y\}_{NS\times 1} = [c]_{NS\times N}\{q\}_{N\times 1} \tag{1}$$

In this model, the response is fully described by a finite number of degrees of freedom (DOFs) q that depend on time/frequency. The dynamic stiffness matrix $Ms^2 + C_s + K$ gives the relation between the response of the model DOFs q and the model loads $[b]\{u\}$. (The dynamic stiffness is assumed symmetric, but extensions to nonsymmetric cases are possible.)

Physical displacements (translations, rotations, stresses, strains, etc.) are called outputs y and are assumed to be linearly related to the DOFs q through output shape matrices c. For example, the matrix c associated with displacement outputs of a displacement-based finite element corresponds to the evaluation of the element shape functions at the considered node.

Similarly, loads (applied forces, aerodynamic or acoustic pressure fields, control forces, gravity, etc.) are represented by the product of time-independent input shape matrices b and time/frequency dependent inputs u.

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Rayleigh–Ritz analyses, which could appropriately be called displacement-based reduction approaches, seek approximate solutions in a reduced subspace corresponding to the range of a rectangular matrix T described by reduced DOFs q_R:

$$\{q_{\text{True}}\}_{N\times 1} \approx [T]_{N\times NR}\{q_R\}_{NR\times 1} \tag{2}$$

The validity of the projection is based on the assumption that all effectively found displacements q of the full-order model have a close approximation in the range of T. Convergence in Rayleigh–Ritz analyzes generally implies the verification of kinematic boundary conditions. In the present case, these conditions are implicitly eliminated from the full-order model (1) and thus verified by the reduction basis T. It is further assumed that the columns of T are independent.

The projection (2) applied to loads and displacements of the full-order model (1) leads to the reduced model (with NR rather than N DOFs)

$$[T^TMTs^2 + T^TCTs + T^TKT]_{NR\times NR}\{q_R\}_{NR\times 1} = [T^Tb]_{NR\times NA}\{u\}_{NA\times 1} \tag{3}$$

$$\{y\}_{NS\times 1} = [cT]_{NS\times NR}\{q_R\}_{NR\times 1}$$

Models are used to compute qualities that characterize the system response. Typical qualities are static responses to fixed loads, stress/strain distributions, modal frequencies, mode shapes, or damped system responses. Full (1) and reduced (3) models can give estimates of the same qualities. The difficulty is to choose the reduction basis T so that, for the qualities of interest, predictions of the full and reduced models are similar.

The published ways to construct reduction bases are so numerous that the search for an exhaustive listing would be futile. In almost all cases, however, the idea is to use solutions to representative subproblems. Typical retained solutions are normal modes of components using different boundary conditions (fixed,[2] free,[3] loaded,[4] etc.), static responses of components to applied loads (often called attachment modes[1]), or to prescribed displacements on boundary DOFs (often called constraint modes[1]), or simply selected DOFs (one then talks of static or Guyan condensation[5]).

CMS methods usually construct bases for substructures and couple them by imposing displacement (generalized kinematic boundary condition) and sometimes force equilibrium[3] (generalized natural boundary condition) constraints on substructure interfaces. These constraints can be taken into account explicitly by finding a reduced basis verifying the constraint or appended in the form of Lagrange multipliers[6] (users of parallel computers have shown a renewed interest in this approach).

III. Principles for an Optimal Selection of Vectors

A. Error Measures and Optimal Projections

A prerequisite for the definition of optimal sets of vectors is the definition of norms on the vector spaces of interest. The usual norm used for structures is the global strain energy linked to the static response to a single load $b(NA = 1)$ or displacement $q(NS = 1)$:

$$\|b\|_{2k} = b^TK_{\text{Flex}}^{-1}b \quad \text{or} \quad \|q\|_{2K} = q^TKq \tag{4}$$

The strain energy is a good norm because it takes into account the physical significance and scaling of individual load or displacement components. The pseudo-inverse K_{Flex}^{-1}, needed in cases with rigid-body modes, is discussed in Sec. III.C.

A basic result of the Rayleigh–Ritz method is that the reduced model (3) gives exact solutions for displacements that lie within the range of T. An obvious way to measure modeling errors is thus to define a projection π of arbitrary vectors q on the range of T and to use the norm of the residual $q - \pi(q)$ to measure the error.

The projection $\pi(q)$ lying in the range of T can be expressed using reduced coordinates q_{PR}, in other words, as the product $\{\pi(q)\} = [T]\{q_{PR}\}$. The reduced coordinates of the projection q_{PR} can be chosen so as to minimize the strain energy of the residual and are then given by

$$\{q_{PR}\} = \arg\min_{q_R}\left[(q - Tq_R)^TK(q - Tq_R)\right] = [T^TKT]^{-1}[T^TKq] \tag{5}$$

In most problems, however, the load is given and one uses reduced models to avoid the cost of computing the exact displacement q. Rather than the displacement projection, it is thus more practical to use the optimal (in the sense of the strain energy norm) projection of a static load b on the basis KT of exactly represented loads. This projection is given by

$$\pi(b) = [KT][T^TKT]^{-1}[T^Tb] \tag{6}$$

Formulas (5) and (6) clearly indicate that optimal projections associated to the strain energy norm correspond to static solutions of the reduced model equations (3). This gives a strong motivation for the use of the strain energy norm, but in some cases one may also consider the kinetic energy norm linked to static loads or displacements defined by

$$\|b\|_{2M} = b^TK_{\text{Flex}}^{-1}MK_{\text{Flex}}^{-1}b \quad \text{or} \quad \|q\|_{2K} = q^TMq \tag{7}$$

Finally, a complete treatment of dynamic problems would imply the definition of a norm with integration over time or frequency to take the time/frequency content of loads/displacements into account. Since such an approach seems difficult to set up for general classes of problems, the present study assumes and verifies in applications that the response y to static inputs u gives indications that are sufficient to evaluate the quality of low-frequency predictions.

B. Multiple Input Cases

In most practical cases, one is interested in the most important responses linked to a set of loads $[b]\{u\}$ (with $NA > 1$). The SVD is a well-known mathematical tool that can be used to rank contributions of a matrix (set of vectors). The SVD of an input shape matrix b takes the general form

$$[b]_{N\times NA} = [U]_{N\times N}[S]_{N\times NA}[V]^T_{NA\times NA} \tag{8}$$

where S has singular values on the diagonal and zeros elsewhere, and the right-hand V and left-hand U singular vectors form orthonormal (unitary) bases.

It is well known that the singular values give a direct indication of the effective rank. Remember, however, that this property is based on two roundness conditions that here would state that all input vectors u of norm 1 and all loads bu of norm 1 are equally representative. One talks of roundness conditions because equally representative inputs and loads lie on spheres. These conditions motivate the selection of orthonormal bases of left- and right-hand singular vectors.

A standard SVD would use Euclidean norms for inputs and loads leading to the two orthonormality conditions $[U^TU] = [I]_{N\times N}$ and $[V^TV] = [I]_{NA\times NA}$. For structures these norms take into account neither variations in the scaling of different DOFs nor inhomogeneities in the component properties. It is thus proposed to use other norms instead.

Loads should be scaled using the strain energy norm (4). The use of this norm does not modify the general form (8) of the SVD, but the left $\hat{U}$ singular vectors now verify the orthonormality condition $[\hat{U}^TK_{\text{Flex}}^{-1}\hat{U}] = [I]_{N\times N}$. In practice, one computes the singular values $\hat{S}$ and right-hand singular vectors $\hat{V}$ by solving the eigenvalue problem

$$\left[b^TK_{\text{Flex}}^{-1}b\right][\hat{V}] = [\hat{V}][\hat{S}]^2 \tag{9}$$

imposing the orthonormality condition $[\hat{V}^T\hat{V}] = [I]_{NA\times NA}$ and deducing the left-hand singular vectors by

$$[\hat{U}] = [b][\hat{V}][\hat{S}]^{-1} \tag{10}$$

In a usual abuse of notation, the basis $\hat{U}$ only contains NA orthonormal vectors and S in Eq. (8) and $\hat{S}$ in Eq. (9) differ by $N - NA$ rows of zeros. This poses no problem, because vectors linked to zero singular values are never used and thus need not be computed.

In some cases, one seeks the decomposition of a set of displacements T that is obtained by the same procedure: solve $[T^T KT][\tilde{V}] = [\tilde{V}][\tilde{S}]^2$ with $[\tilde{V}]^T[\tilde{V}] = [I]$ and compute $[\tilde{U}] = [T][\tilde{V}][\tilde{S}]^{-1}$. If $T = K_{\text{Flex}}^{-1}b$, $\hat{S} = \tilde{S}$ and $\hat{V}^T = \tilde{V}$.

Input scaling must also be considered. Appropriate scaling depends, however, on the choice of loads/displacements, and examples will be considered in Sec. IV.

C. Treatment of Rigid-Body Modes

The full-order model often has a number of rigid-body modes ϕ_{Rig} such that $K\phi_{\text{Rig}} = 0$ (the range of ϕ_{Rig} is the null space of K). When using loads rather than displacements in the methods presented earlier, one thus needs a proper definition of the static response.

It is well known (see Ref. 7, for example) that, for symmetric positive definite mass and symmetric semidefinite stiffness matrices, there exists a basis of mass normalized normal modes ϕ that simultaneously diagonalize the mass and stiffness matrices:

$$[\phi]^T[M][\phi] = [I] \quad \text{and} \quad [\phi]^T[K][\phi] = [\Omega^2] = \text{diag}(\omega_j^2) \tag{11}$$

with a separation $\phi = [\phi_{\text{Rig}} \;\; \phi_{\text{Flex}}]$ into rigid-body (normal modes with null frequencies) and flexible (nonzero frequencies) modes. The static flexible responses of a system with rigid-body modes are defined as the response associated with the flexible normal modes:

$$[T_{\text{Flex}}]_{N\times NA} = \left[K_{\text{Flex}}^{-1}\right][b]_{N\times NA} = \sum_{j\in\{\text{Flex}\}} \left[\phi_j \omega_j^{-2} \phi_j^T\right][b]_{N\times NA} \tag{12}$$

From the mass orthogonality condition in Eq. (11), one easily deduces that loads lying within the range of $M\phi_{\text{Rig}}$ have no flexible response. Equivalently, the responses and flexible responses to loads within the subspace that is orthogonal to $M\phi_{\text{Rig}}$ are the same. The framework defined in Secs. III.A and III.B is thus applied to the load subspace orthogonal to $M\phi_{\text{Rig}}$ by computing the projection b_{Flex} of arbitrary loads b on this subspace:

$$[b_{\text{Flex}}] = [b] - [M\phi_{\text{Rig}}]\left[\phi_{\text{Rig}}^T M\phi_{\text{Rig}}\right]^{-1}\left[\phi_{\text{Rig}}^T b\right] \tag{13}$$

Similarly, flexible responses lie within the subspace mass orthogonal to ϕ_{Rig}. The framework defined in Secs. III.A and III.B is thus applied to the displacement subspace that is mass orthogonal to ϕ_{Rig} by computing the projection T_{Flex} of arbitrary displacements T on this subspace:

$$[T_{\text{Flex}}] = [T] - [\phi_{\text{Rig}}]\left[\phi_{Rig}^T M\phi_{\text{Rig}}\right]^{-1}\left[\phi_{\text{Rig}}^T MT\right] \tag{14}$$

The relations (13) and (14) indicate that the flexible response to a static load $[K_{\text{Flex}}^{-1}][b]$ can be computed in three steps that do not require knowledge of the flexible modes. One computes the flexible load projection (13), the static response to this load with an isostatic constraint constraining rigid-body motion, and finally the projection (14) of the resulting displacement (see more details in Ref. 7).

The need for a specific treatment of rigid-body modes comes from the use in the proposed methods of responses to static loads. For many applications, such as time integration, frequency response function computations, and reduction of damping models, rigid-body modes pose no specific problems, and their presence can be totally ignored. Similarly, bases of real vectors are used here for the reduction of damped models. Such reductions clearly remain valid for systems where damping results in the existence of complex modes, and specific issues[8] need not be addressed here.

IV. Proposed Applications

A. Reduction Using Singular Vectors

A general problem in model reduction applications is the determination of an optimal basis of reduction. This section shows how the general framework proposed in Sec. III can be applied to select a subset of significant contributions in a given basis of retained displacements T.

For an arbitrary basis T there is no reason for the reduced DOFs q_R to be similarly scaled. A proper use of the SVD thus implies the need to define an appropriate norm on the DOFs. A simple possibility is to define the right-hand singular vectors $\bar{V}$ as a solution of the reduced eigenvalue problem

$$[T^T KT][\bar{V}] = [T^T MT][\bar{V}][\bar{S}]^2 \tag{15}$$

and to use the orthonormality condition associated to the kinetic energy norm (7) $\bar{V}^T T^T MT\bar{V} = I$. The left-hand singular vectors are then constructed by

$$[\bar{U}] = [T][\bar{V}][\bar{S}]^{-1} \tag{16}$$

which verifies the orthonormality condition $\bar{U}^T K\bar{U} = I$ based on the strain energy norm (4).

This SVD is based on strain energy at equal kinetic energy levels, because responses with low strain energies are more likely to have high amplitudes. The most important contributions of T are thus given by the left-hand singular vectors in $\bar{U}$ associated to the lowest singular values in $\bar{S}$. An equivalent approach would permute the use of the strain and kinetic energies. In this case, the largest singular values would be kept.

Note that the eigenvalue problems associated with the reduced model (3) and defining the singular values in Eq. (15) are identical, so that SVD and the traditional modal truncation result in the same choice of vectors. The originality of the proposed result lies in the fact that the reduced eigenvalue problem (15) now appears as appropriate to select the most important contributions in any set of displacement vectors (and not only the full set considered in modal truncation).

In the Craig–Bampton CMS method, component coupling is obtained through a set of Ritz vectors defined over the whole structure and corresponding to unit static displacements of the interface DOFs (the combination of substructure constraint modes that is continuous at interfaces). The general form of the Ritz basis used in this method is

$$\begin{bmatrix} T_{C1} \\ \vdots \\ T_{Cn} \\ T_I \end{bmatrix} = \begin{bmatrix} -K_{CC1}^{-1} & K_{CI1} & \phi_{C1} & 0 & 0 \\ \vdots & & 0 & \ddots & 0 \\ -K_{CCn}^{-1} & K_{CIn} & 0 & 0 & \phi_{Cn} \\ I & & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

where interface DOFs correspond to the block of DOFs I and component modes ϕ_{Ci} with nonzero components only on interior DOFs Ci of component i. The fact that there are as many of these vectors as there are interface DOFs I is a major drawback, and further reduction is strongly desirable.

The singular values of the matrix of constraint modes are defined by the eigenvalue problem (15), where one easily verifies that $T^T KT$ and $T^T MT$ are the stiffness and mass matrices corresponding to a static condensation[5] onto the interface DOFs. The singular vectors thus appear as modes of the model condensed on the interface DOFs with the lowest frequency modes retained.

For methods based on free or loaded interface normal modes,[3,4] continuity is generally achieved by combining normal modes and responses to static interface loads or deformations (correction modes). A reduction of the set of retained correction modes will thus imply the need to use an incompatible model where interface continuity is not fully verified, which may give good results but poses problems that are beyond the scope of this paper.[9]

B. Correction of a Given Model

A second typical problem is the evaluation of errors linked to predicted normal modes. The proposed approach defines imbalance loads, which give good indications of the error for a set of normal modes, and corrects the initial reduced model by retaining significant components of the SVD of the flexible response to the imbalance loads.

For a prediction $[\phi_R]_{N\times NP}$ of the first NP modes of the full model, one considers the loads b_{jM} needed to balance the full-order equations at the computed resonance (for exact modes these loads would be null):

$$\{b_{jM}\} = \left[-M\omega_{jR}^2 + K\right]\{\phi_{jR}\} \tag{18}$$

The imbalance loads $\{b_{jM}\}$ give a very good indication of the loads that need to be accurately represented to obtain a good prediction of modes (a motivation for this result can be found in studies of inverse iteration methods for partial eigenvalue solvers[7]). For mass normalized modes $\{[\phi_R^T M \phi_R] = [I]_{NP \times NP}\}$, one can assume that the columns of the input shape matrix $[b_M] = [b_{1M} \;\cdots\; B_{NP\,M}]$ are similarly scaled and use the decomposition (9) and (10).

The largest singular value gives a direct indication of the quality of a model. Furthermore, the vectors associated to the largest singular values are very good candidates for additional vectors to be retained as corrections to the initial reduced model.

In practice, this approach can be efficiently used as follows. A first estimation of the normal modes is done using a small reduction basis (see Sec. V.C for examples). The static flexible response (12) to the imbalance loads $[b_M]$ is then computed leading to imbalance modes. Significant contributions of the imbalance modes are determined by computing their stiffness singular value decomposition (9) and (10). The initial reduced model is then corrected by adding contributions with significant singular values to the initial reduction basis and computing the response of the new reduced model.

V. Illustration for an Assembly of Two Plates

A. Definition of the Example

The approaches proposed in this paper will be illustrated for the case of two square plates perfectly connected at a right angle. The properties are $E = 210e9\,\text{N/m}^2$, $\nu = 0.3$, $\rho = 7800\,\text{kg/m}^3$, and $L = l = 1$ m. The first plate thickness is 0.05 m. Two thicknesses (0.005 and 0.05 m) are considered for the second plate thickness. Computations are done in MATLAB,[10] using for each plate a regular 9×9 mesh (shown in Fig. 1) of 81 simple QUAD4 thin plate elements.

The considered reduced models are built to predict the first 20 modes (which include 6 rigid-body modes in all considered cases) of the structure. The maximum relative error on the first 20 modal frequencies,

$$\alpha = \max_{j=1,20}(|\omega_{jR} - \omega_j|/\omega_j) \tag{19}$$

where ω_{jR} is the approximate resonance frequency and ω_j the numerically exact value associated to the 1140-DOF model, is taken as the classical error measure.

B. Reduction of Interface Models

An evaluation of the method proposed is Sec. IV.A for the selection of interface modes was performed, and results are summarized in Table 1. The two extreme plate thicknesses (0.005 and 0.05 m) are considered. The models contain 20 fixed interface modes (10 for each plate in the equal thickness configuration, 19 of the thin plate and 1 of the thick plate in the thick/thin configuration). The basis of fixed interface modes is complemented by constraint modes that correspond to unit displacements of the interface DOFs [this is a model of the form (17)].

The SVD (15) and (16) of the matrix of constraint modes was first computed. In the present case, the system has free-free boundary conditions. One expects, and verifies in practice, that the singular vectors with the lowest six singular values in the decomposition of the matrix of constraint modes correspond to six global rigid-body modes. Although the maximum frequency errors are significant (model B of Table 1) when only those six modes are retained, a detailed analysis clearly shows that a majority of the first 20 modes are already well predicted.

The predictions made with 1, 2, and 3 interface modes (models C–E) clearly show that in both configurations most of the error can be eliminated with just two flexible interface modes. The shape of these two modes depends on the configuration, but one finds that their frequencies [singular values of Eq. (15)] are within the bandwidth of interest, whereas the third mode is higher ($9.0\;10^{+3}$ rad/s for mode 20 at $4.9\;10^{+3}$ rad/s in the equal thickness configuration and $1.5\;10^{+4}$ rad/s for $1.0\;10^{+3}$ rad/s in the thick/thin configuration). Traditional frequency separation criteria are thus applicable in the present case.

The SVD (15) uses mass normalized modes to obtain a proper scaling of the retained vectors. When the basis of constraint modes is directly decomposed [replacing $T^T M T$ by an identity matrix in Eq. (15)], the roundness hypothesis on the representativity of inputs is not well taken into account. The results are not as good but are not significantly worse (model F). Another simplification would be to approximate the mass distribution by replacing M by an identity matrix in the decomposition (15). The results (model G) are only slightly worse than those obtained with the decomposition (15) (a significant part of the 0.62% in the thick/thin plate configuration is a result of numerical conditioning problems linked to the selected finite element). Note that a mathematical study of convergence with this simplification of the mass matrix can be found in Ref. 11.

The results shown in Table 1 clearly demonstrate that the proposed approach for the construction of vectors describing the most important interface deformations works for substructures with both similar and very different properties. The level of reduction achieved is such that the use of fixed interface modes becomes practical in cases where the total number of DOFs in the reduced model should be very limited (in aeroelasticity or structural optimization[12] studies, for example). The interface modes can also be constructed using local problems,[9] thus avoiding the need to compute a full basis of constraint modes. When using free or loaded interface modes (Rubin's[3] method, for example), the full basis of constraint modes is used to achieve continuity at the interface, but for coupled predictions residual interface modes are often eliminated a posteriori. Uses of the method proposed here are thus not obvious.

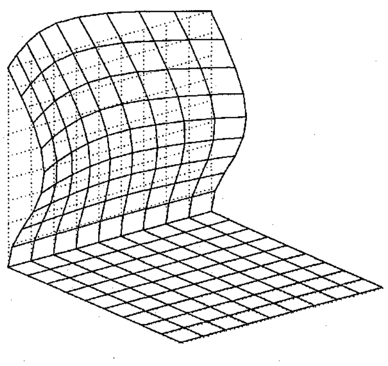

Fig. 1 Third flexible mode of the plate assembly (plate thicknesses equal to 0.005 and 0.05 m).

C. Model Correction

The model correction method proposed in Sec. IV.B was tested for the two-plate example. This method is generally applicable to initial predictions of any reduced model, and three initial vector sets based on fixed, free, and loaded interface normal modes were considered. Associated results are shown in Table 2 for both the equal thickness and thick/thin plate configurations.

Table 1 Maximum relative frequency errors α for two configurations and different reduction bases

Model	Number of modes	α for 0.05 and 0.005 m, %	α for 0.05 and 0.05 m, %
A: Craig–Bampton	80	0.43	0.95
B: 20 fixed, 6 rigid	26	27.12	57.20
C: 20 fixed, 6 rigid, 1 interface	27	0.47	13.41
D: 20 fixed, 6 rigid, 2 interface	28	0.47	0.99
E: 20 fixed, 6 rigid, 3 interface	29	0.47	0.99
F: $20 + 6 + 3\,(T^T M T \rightarrow I)$	29	0.55	1.53
G: $20 + 6 + 3\,(M \rightarrow I)$	29	0.62	0.97

Table 2 Model accuracy (frequency error α and maximum singular value s_{Max} of the imbalance loads) for the two configurations and different model reduction methods

		0.05 and 0.05 m		0.05 and 0.005 m	
Model	DOF	α, %	s_{Max}	α, %	s_{Max}
0: Numerical exact modes	1140	0.00	$2.3\ 10^{-5}$	0.00	$4.6\ 10^{-2}$
A: Craig–Bampton	80	0.95	$5.3\ 10^{+2}$	0.43	$3.7\ 10^{+1}$
B: fixed, six rigid	26	57.20	$5.5\ 10^{+3}$	27.12	$6.1\ 10^{+1}$
C: fixed, six rigid, five interface	31	0.99	$5.4\ 10^{+2}$	0.47	$3.8\ 10^{+2}$
D: fixed, six rigid, five correction	31	0.35	$3.2\ 10^{+2}$	0.05	$1.2\ 10^{+2}$
E: free interface method	26	30.76	$2.8\ 10^{+3}$	26.87	$7.0\ 10^{+2}$
F: free, five correction	31	0.05	$1.2\ 10^{+2}$	0.06	$6.8\ 10^{+1}$
G: loaded interface method	26	5.71	$9.3\ 10^{+2}$	0.00	$8.4\ 10^{+0}$
H: loaded, five correction	31	0.39	$3.2\ 10^{+2}$	0.00	$3.9\ 10^{-1}$

With only five correction modes corresponding to the SVD of 20 imbalance loads, Eq. (18), the corrected models (D, F, and H) all produce good results and achieve significant performance improvements over the results obtained with traditional model reduction methods (models B, C, E, and G).

Models A–D use fixed interface modes. Models A–C are described in Sec. V.B, and model D is the correction of model B with five imbalance modes retained. The construction of model D implies the computation of the imbalance loads, Eq. (18), and the associated static flexible response. These computations can be performed without assembling the full-order model, and their cost is similar to if not lower than that linked to the computation of the constraint modes used in models A and C. The proposed prediction/correction approach thus appears as a practical alternative to the Craig–Bampton method.

For free or loaded interface modes, the exact or approximate verification of continuity conditions at the interface usually introduces very significant constraints (and thus leads to poor results) unless interface deformation modes are added to the bases. Model E is obtained by constructing a basis of free interface modes, complemented by constraint modes, enforcing the displacement continuity constraint, and retaining the first 26 modes of the resulting model (this approach is very close to Rubin's[3] method). In both configurations, the correction (model F) strongly improves results.

Model G is constructed by computing modes of one substructure when the DOFs of the other substructure are statically condensed[5] onto the interface. Computationally, this approach is quite efficient, because the static expansion directly leads to modes defined over the whole structure and continuous at the interface. Note that the branch mode method[4] introduced the idea that static expansions could be used to obtain continuous system modes based on arbitrary component modes. Model G gives almost exact results in the thick/thin plate configuration. This can be explained by the fact that dynamic interactions between the two very dissimilar plates are negligible. For the correction (model H), the maximum singular value of the imbalance loads still indicates that the correction improves the accuracy.

In all cases, the maximum singular value of the decomposition of the imbalance loads shows a strong correlation with the maximum frequency error used otherwise and thus appears as a good measure to compare the quality of two models with no need to know the full-order solution.

There is no indication, however, of what values should be expected for a good model, and values encountered in practice change for different configurations (roughly a factor 10 between the two configurations studied here). The definition of such reference values is a very important research topic, since it would allow the use of the maximum singular value of the decomposition of the imbalance loads as an absolute measure of model error.

VI. Conclusions

The strain energy norm and the associated SVD have been shown to be very practical tools for use in component mode synthesis and related model reduction methods. Among many possible applications of the proposed formalism, two methods for the optimal selection of interface modes and for the correction of reduced models have been analyzed, and their effectiveness has been illustrated in an example.

The optimal selection method allows significant reduction of model size in the Craig–Bampton method, which may extend the applicability of this method for cases where the total number of DOFs must be limited. For applications in design, model updating, and nonlinear simulations, reduced models can be constructed that include parametrized representations of property variations. The number of loading cases that are then considered becomes extremely significant, and vector selection methods based on the proposed SVD are expected to play a very significant role.

The correction method uses static responses of the full-order model but allows very significant improvements of normal mode predictions at a relatively low cost. The SVD of the imbalance loads provides a direct measure of the model error, which can currently be used for comparisons of different models but lacks reference values for a use as an absolute measure of model quality. This measure allows comparisons with no need to compute an exact solution and can also be shown to be adapted for cases with closely spaced modes that may recombine easily.

The interface modes and imbalance loads used here do not take external inputs and their frequency content into account. A proper treatment of forced dynamic responses would imply the definition of representative time/frequency content for applied forces and an integration over time/frequency, in other words the resolution of a Lyapunov equation. Assuming that numerical problems linked to the large size of the considered models can be resolved, such approaches will certainly lead to useful results.

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